

On the Stueckelberg Like Generalization of General Relativity

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Abstract

We first consider the Klein-Gordon equation in the 6-dimensional space $M_{2,4}$ with signature $+- - - - +$ and show how it reduces to the Stueckelberg equation in the 4-dimensional spacetime $M_{1,3}$. A field that satisfies the Stueckelberg equation depends not only on the four spacetime coordinates x^μ , but also on an extra parameter τ , the so called evolution time. In our setup, τ comes from the extra two dimensions. We point out that the space $M_{2,4}$ can be identified with a subspace of the 16-dimensional Clifford space, a manifold whose tangent space at any point is Clifford algebra $Cl(1,3)$. Clifford space is the space of oriented r -volumes, $r = 0, 1, 2, 3$ associated with the extended objects living in $M_{1,3}$. We consider the Einstein equation that describe a generic curved space $M_{2,4}$. The metric tensor depends on six coordinates. In the presence of an isometry given by a suitable Killing vector field, the metric tensor depends on five coordinates only, which include τ . Following the formalism of the canonical classical and quantum gravity, we perform the $4 + 1$ decomposition of the 5-dimensional general relativity and arrive, after the quantization, at a generalized Wheeler-DeWitt equation for a wave functional that depends on the 4-metric of spacetime, the matter coordinates, and τ . Such generalized theory resolves some well known problems of quantum gravity, including “the problem of time”.

1 Introduction

1.1 The problem of time in quantum gravity

Despite being a very successful theory at the classical level, general relativity has turned out to be problematic when attempted to be consistently quantized. Amongst others, there is the so called ‘problem of time’ (for a recent review see [1]). This can be seen if we perform the canonical quantization. If we start from the Einstein-Hilbert action, and perform the $1 + 3$ ADM decomposition of spacetime, $M_{1,3} = \mathbb{R} \times \Sigma$, then the action of general relativity can be cast into the ‘phase space’ form [2, 3]

$$I[q_{ij}, p^{ij}, N, N_i] = \frac{1}{16\pi G} \int dt d^3x [p^{ij} \dot{q}_{ij} - N H(q_{ij}, p^{ij}) - N_i H^i(q_{ij}, p^{ij})]. \quad (1)$$

Here q_{ij} , $i, j = 1, 2, 3$, is a 3-metric on a space hypersurface Σ , and p^{ij} is the corresponding canonically conjugate momentum, whilst N and N_i are, respectively, laps

and shift functions having the role of Lagrange multipliers leading to the constraints

$$\mathcal{H}(q_{ij}, p^{ij}) \approx 0, \quad \text{and} \quad \mathcal{H}^i(q_{ij}, p^{ij}) \approx 0, \quad (2)$$

which are associated with the diffeomorphism invariance of the original Einstein-Hilbert action. The Hamiltonian is a linear combination of constraints and the evolution is a pure gauge. There is no physical evolution time in such a theory.

Upon quantization, the above constraints become the wave functional equations. For instance, the first constraint becomes the Wheeler-DeWitt equation

$$\mathcal{H}(q_{ij}, -i\frac{\delta}{\delta q_{ij}}) \Psi[q_{ij}] = 0 \quad (3)$$

whilst the second constraints become

$$\mathcal{H}^i(q_{ij}, -i\frac{\delta}{\delta q_{ij}}) \Psi[q_{ij}] = 0. \quad (4)$$

We see that in quantum theory there is no spacetime, but only space Σ , because the wave function(al) depends only on 3-geometry, represented by q_{ij} . Thus, in addition to the absence of an external time, we have also the problem of the disappearance of spacetime.

1.2 A possible remedy: the Stueckelberg theory

In the Stueckelberg theory [4], besides the four spacetime coordinates x^μ , there is an extra parameter τ . The coordinate $x^0 \equiv t$ is not the ‘evolution parameter’. The evolution parameter is τ .

In quantum theory of a ‘point particle’, the wave function is

$$\psi(\tau, x^\mu), \quad (5)$$

and is normalized according to $\int d^4x \psi^* \psi = 1$. We will show how τ arises from extra two dimensions, one space like and one time like dimension.

Then we will show that ‘extra dimensions’ need not be the ‘true’ extra dimensions, i.e., some extra dimensions in addition to four spacetime dimensions, but can be associated with the space of matter configurations. A particular case of such configuration space is Clifford space, a manifold whose tangent space at any point is the Clifford algebra $Cl(1,3)$. Clifford space is the space of oriented r -volumes, $r = 0, 1, 2, 3$ associated with the extended objects living in $M_{1,3}$. In this paper we focus our attention to a 6-dimensional subspace, $M_{2,4}$, of Clifford space. We consider the Einstein equations that describe a generic curved space $M_{2,4}$. Then we perform the ADM-like 1+4 decomposition of a 5-dimensional subspace $M_{2,3}$ of $M_{2,4}$, our argument being that the additional dimension can be neglected in the presence of an isometry given

by a suitable Killing vector field, because then the metric tensor depends on five coordinates only.

We will show how in the quantized theory the problems of time and of spacetime disappear in such generalized theory. The latter problem does not occur, because the wave functional now depends on spacetime 4-geometry, represented by the metric $g_{\mu\nu}$. The problem of time we resolve by adding a suitable matter part to the action.

2 Klein-Gordon equation in 6D

Let us consider the action for the Klein-Gordon field in 6-dimensions:

$$I[\phi, \phi^*] = \int d^6x \partial_M \phi^* \partial^M \phi \quad (6)$$

where $\phi = \phi(x^M)$, $M = 0, 1, 2, 3, 5, 6$. Let us split the index M into a 4-dimensional part and the part due to the extra two dimensions according to $M = (\mu, \bar{M})$, $\mu = 0, 1, 2, 3$, $\bar{M} = 5, 6$, and let us assume that the metric has the following form:

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (7)$$

The latter metric can be transformed into

$$G'_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

which is the pseudo euclidean metric with signature $(+ - - - - +)$. By inserting the metric (7) into the action (6), we obtain

$$I[\phi, \phi^*] = \int d^6x (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \partial_\tau \phi^* \partial_\lambda \phi - \partial_\lambda \phi^* \partial_\tau \phi), \quad (9)$$

where we have denoted $x^5 \equiv \tau$, $x^6 \equiv \lambda$.

Taking the ansatz

$$\phi = e^{i\Lambda\lambda} \psi(\tau, x^\mu), \quad (10)$$

where Λ is a constant, we have

$$I[\psi, \psi^*] = \int d\tau d^4x [\partial_\mu \psi^* \partial^\mu \psi + i\Lambda (\psi^* \partial_\tau \psi - \partial_\tau \psi^* \psi)], \quad (11)$$

which is the well known Stueckelberg action. We have omitted the integration over λ , because it gives a constant which can be absorbed into the expression.

Considering the corresponding equations of motion, we find that from a massless Klein-Gordon equation in 6D

$$\partial^M \partial_M \phi = 0 \quad (12)$$

we obtain the Stueckelberg equation

$$i \partial_\tau \psi = \frac{1}{2\Lambda} \partial^\mu \partial_\mu \psi \quad (13)$$

The constant Λ comes from the 6th dimension $x^6 \equiv \lambda$. More precisely, Λ is an eigenvalue of the canonical momentum conjugate to λ . By ansatz (10), coordinate λ is eliminated from the action, whilst the eigenvalue Λ remains.

To sum up, if the signature of two extra dimensions is $(-+)$, and if we work in the light cone coordinates, then we obtain the Stueckelberg equation for a wave function that depends on τ and x^μ . Notice that, because τ is a ‘light cone’ coordinate, we have the first derivative of ψ with respect to τ .

2.1 More formal considerations: Point particle in 6D and its quantization

Let us consider a classical action for a point particle in 6-dimensional space:

$$I[X^M] = M \int d\sigma (\dot{X}^M \dot{X}_M)^{1/2}, \quad (14)$$

where $M = 0, 1, 2, 3, 5, 6$. Here σ is a parameter, denoting a point on the worldline, and $\dot{X}^\mu = dX^\mu/d\sigma$.

An equivalent action is a functional of the coordinates X^M , the canonically conjugate momenta P_M , and a Lagrange multiplier α :

$$I[X^M, P_M, \alpha] = \int d\sigma \left(P_M \dot{X}^M - \frac{\alpha}{2} (P_M P^M - M^2) \right). \quad (15)$$

Varying the latter action with respect to P_M , we obtain the relation between velocities and momenta, $\dot{X}^M = \alpha P^M$.

If we split the coordinates according to

$$X^M = (X^\mu, X^{\bar{M}}), \quad \bar{M} = 5, 6, \quad (16)$$

and express four momenta in terms of velocities, $P^\mu = \dot{X}^\mu/\alpha$, then the action (15) becomes

$$I[X^\mu] = \int d\sigma \left(\frac{\dot{X}^\mu \dot{X}_\mu}{2\alpha} + P_{\bar{M}} \dot{X}^{\bar{M}} - \frac{\alpha}{2} (P_{\bar{M}} P^{\bar{M}} - M^2) \right). \quad (17)$$

The second term in the latter action can be omitted, because by partial integration it can be transformed into the form

$$\int d\sigma \left(\frac{d}{d\sigma} (P_{\bar{M}} X^{\bar{M}}) - \dot{P}_{\bar{M}} X^{\bar{M}} \right), \quad (18)$$

and if we use the equations of motion, $\dot{P}_{\bar{M}} = 0$, then only the total derivative term remains.

The third term in eq. (17) can be rewritten in terms of the 4D mass, m . Namely, by varying (15) we obtain the mass shell constrain in 6D:

$$\delta\alpha : G^{MN} P_M P_N - M^2 = 0, \quad (19)$$

which can be decomposed according to

$$g^{\mu\nu} P_\mu P_\nu + G^{\bar{M}\bar{N}} P_{\bar{M}} P_{\bar{N}} - M^2 = 0. \quad (20)$$

From $M^2 = P^M P_M = P^\mu P_\mu + P_{\bar{M}} P^{\bar{M}}$, we have

$$m^2 = M^2 - P_{\bar{M}} P^{\bar{M}}, \quad (21)$$

where $m^2 \equiv P^\mu P_\mu$. If 6D mass M is equal to zero, then the 4D mass is due to the 5th and the 6th component of momentum only:

$$m^2 = G^{\bar{M}\bar{N}} P_{\bar{M}} P_{\bar{N}} = 2P_5 P_6. \quad (22)$$

So, from eq. (17), using (18), and (21), we obtain the well known Howe-Tucker action for a massive particle in 4-dimensional spacetime:

$$I[X^\mu] = \frac{1}{2} \int d\sigma \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\alpha} + \alpha m^2 \right). \quad (23)$$

Upon quantization, the classical constraint (19) becomes the Klein-Gordon equation

$$(G^{MN} \hat{P}_M \hat{P}_N - M^2) \phi = 0, \quad (24)$$

where $\hat{P}_M = -i\hbar\partial/\partial X^M$ is the momentum operators. We will use unit in which $\hbar = c = 1$, and write $\partial_M \equiv \partial/\partial X^M$.

We can decompose eq. (24) into a 4D and a 2D part:

$$(-g^{\mu\nu} \partial_\mu \partial_\nu - G^{\bar{M}\bar{N}} \partial_{\bar{M}} \partial_{\bar{N}} - M^2) \phi = 0, \quad (25)$$

which gives

$$(-g^{\mu\nu} \partial_\mu \partial_\nu + 2\partial_5 \partial_6 - M^2) \phi = 0. \quad (26)$$

By the ansatz

$$\phi = e^{iP_6 x^6} \psi(x^5, x^\mu) \quad (27)$$

and by denoting $x^5 \equiv \tau$, $P_6 \equiv \Lambda$, eq. (26) gives

$$i \frac{\partial}{\partial \tau} \psi = \frac{1}{2\Lambda} (g^{\mu\nu} \partial_\mu \partial_\nu + M^2) \psi \quad (28)$$

If, in particular, 6D mass M is zero, then we have the usual Stueckelberg equation (13).

We have seen that the Stueckelberg equation in which the wave function depends not only on four spacetime coordinates x^μ , but also on an extra parameter τ (evolution parameter), is embedded in the 6D theory with one time-like and one space-like extra dimension:

6D space $M_{2,4}$ signature $+- - - - +$	\longrightarrow	Stueckelberg theory in $M_{1,3}$ with invariant evolution parameter τ
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At this point it is interesting to observe, that an extra time like and an extra space like dimension are necessary in the “two time” physics [5], based on the extended phase space action that is invariant under local $\text{Sp}(2)$ transformations. Our action (15), with $M = 0$, is a particular case of a more general action, considered in refs. [5].

A question arises as to what is a physical meaning of the extra two dimensions. This will be discussed in next section.

3 The space $M_{2,4}$ as a subspace of Clifford space

Clifford space, C , is the space of oriented r -volumes, $r = 0, 1, 2, 3$, associated with extended objects living in spacetime $M_{1,3}$. ‘Extra dimensions’ are due to the fundamental extended nature of physical objects. The concept of Clifford space has been discussed in refs. [6]–[15]. Here we will exploit the fact that the space $M_{2,4}$, used in previous section, can be identified with a subspace of C .

3.1 Clifford space: a quenched configuration space of extended objects—branes

Strings and branes have infinitely many degrees of freedom. But at first approximation we can consider just the center of mass, X^μ , $\mu = 0, 1, 2, 3$. Next approximation is in considering the holographic coordinates, $X^{\mu\nu}$, of the oriented area enclosed by the string. We may go even further and search for eventual thickness of the object. If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom, $X^{\mu\nu\rho}$.

In general, for an extended object in $M_{1,3}$, we have 16 coordinates

$$x^M \equiv x^{\mu_1 \dots \mu_r}, \quad r = 0, 1, 2, 3, 4. \quad (29)$$

They are the projections of r -dimensional volumes (areas) onto the coordinate planes. Oriented r -volumes can be elegantly described by Clifford algebra [14, 15]. Although branes have infinitely many degrees of freedom, we can sample them by a finite set of coordinates X^M , that denote position in a 16D manifold, called Clifford space C , whose tangent space at any point X^M is the Clifford algebra $Cl(1, 3)$.

Instead of the usual relativity formulated in spacetime in which the interval is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (30)$$

let us consider the theory in which the interval is extended to Clifford space:

$$dS^2 = G_{MN} dx^M dx^N \quad (31)$$

where $dx^M \equiv dx^{\mu_1 \dots \mu_r}$, $r = 0, 1, 2, 3, 4$.

Coordinates of Clifford space can be used to model extended objects, whatever they are. The latter coordinates, the so called ‘polyvector coordinates’, are a generalization of the concept of center of mass [14]. Instead of describing extended objects in “full detail”, we can describe them in terms of the center of mass, area and volume coordinates. Therefore, Clifford space is a quenched configuration space for extended objects [16]. In particular, extended objects can be fundamental branes.

The line element (31) can be written as the scalar product of the Clifford number

$$dX = dx^M \gamma_M \equiv dx^{\mu_1 \mu_2 \dots \mu_r} \gamma_{\mu_1 \mu_2 \dots \mu_r}, \quad r = 0, 1, 2, 3, 4 \quad (32)$$

with its reverse dX^\dagger :

$$dS^2 \equiv |dX|^2 \equiv dX^\dagger * dX = dx^M dx^N G_{MN} \equiv dx^M dx_M. \quad (33)$$

The metric is given by the scalar product of the basis Clifford numbers:

$$G_{MN} = \gamma_M^\dagger * \gamma_N \equiv \langle \gamma_M^\dagger \gamma_N \rangle_0. \quad (34)$$

Reversion, denoted by \dagger , is an operation that reverses the order of vectors in a Clifford product: $(\gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_r})^\dagger = \gamma_{\mu_r} \dots \gamma_{\mu_2} \gamma_{\mu_1}$. In flat Clifford space, $\gamma_M \equiv \gamma_{\mu_1 \mu_2 \dots \mu_r}$ is the wedge product of basis vectors, $\gamma_M = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \dots \wedge \gamma_{\mu_r}$, at every point of C .

With the definition (34) of the metric, signature of C is $(8, 8)$. Therefore, $M_{2,4}$ is a subspace of C .

3.2 Dynamics

The following action generalizes the action for a point particle of the ordinary special relativity:

$$I = \int d\sigma (\eta_{MN} \dot{X}^M \dot{X}^N)^{1/2}, \quad (35)$$

where σ is an arbitrary continuous parameter. From the latter action we obtain the following equations of motion:

$$\ddot{X}^M \equiv \frac{d^2 X^M}{d\sigma^2} = 0 \quad (36)$$

Here η_{MN} is the analogue of Minkowski metric with signature (8, 8).

Since X^M are interpreted as r -volume coordinates, the equations of motion (36) imply that the volume (in particular the area) changes linearly with σ . If X^M sample a brane, then the above dynamics can only hold for a tensionless brane. For a brane with tension one has to introduce curved Clifford space and generalize eqs. (35), (36) to arbitrary metric with non vanishing curvature [9, 10, 11].

A worldline $X^M(\sigma)$ in C represents the evolution of a ‘thick particle’ in spacetime $M_{1,3}$. In C we have a line, a worlline $X^M(\sigma)$, whilst in spacetime $M_{1,3}$, we have a thick line whose centroid line is $X^\mu(\sigma)$. It describes a thick particle, i.e., and extended object, in spacetime. The thick particle can be an aggregate of p -branes for various $p = 0, 1, 2, \dots$. But such interpretation is not obligatory. A thick particle may be a conglomerate of whatever extended objects that can be sampled by ‘polyvector’ coordinates $X^M \equiv X^{\mu_1 \mu_2 \dots \mu_r}$.

4 Einstein’s equations in $M_{2,4}$

A subspace of Clifford space C is $M_{2,4}$. Let x^M , $M = 0, 1, 2, 3, 5, 6$ be coordinates, and $G_{MN} = G_{MN}(x^M)$ a metric tensor in $M_{2,4}$. The Einstein-Hilbert action in the presence of a point like source¹ read

$$I[X^M, G_{MN}] = M \int d\sigma (\dot{X}^M \dot{X}^N G_{MN})^{1/2} + \frac{1}{16\pi\mathcal{G}} \int d^6x R^{(6)} \quad (37)$$

If we vary the latter action with respect to $X^M(\sigma)$, we obtain the geodesic equation,

$$\frac{1}{\sqrt{\dot{X}^2}} \frac{d}{d\sigma} \left(\frac{\dot{X}^M}{\sqrt{\dot{X}^2}} \right) + \Gamma_{JK}^M \frac{\dot{X}^J \dot{X}^K}{\dot{X}^2} = 0, \quad (38)$$

¹ In our interpretation of the space $M_{2,4}$ as a subspace of Clifford space C , which is a configuration space associated with an extended object, a point like source in $M_{2,4}$ is a thick source in 4D spacetime $M_{1,3}$.

and if we vary it with respect to $G_{MN}(x^M)$, we obtain the Einstein equations,

$$R^{MN} - \frac{1}{2}G^{MN}R = 8\pi\mathcal{G} \int d\sigma \delta(\xi - \mathcal{X}(\sigma))\dot{\mathcal{X}}^{\mathcal{M}}\dot{\mathcal{X}}^{\mathcal{N}}. \quad (39)$$

At this point let us mention that, according to a general consensus, the Einstein equations with a point like source have no solution, because a solution in the vacuum around a source is the black hole solution with a horizon. However, a penetrating discussion about this issue can be found in ref. [17]. The fact that a point like source action (37) is problematic could mean that the fundamental objects are not point particles, but branes. Instead of a worldline $X^M(\sigma)$, one should take a brane $X^M(\sigma^a)$ as a source. From the point of view of the 16D Clifford space C , or its 6D subspace $M_{2,4}$, the latter object is infinitesimally thin in the directions transverse to the brane. But from the point of view of 4D spacetime $M_{1,3} \subset M_{2,4} \subset C$ we have a *thick brane*.

Leaving such intricacies aside, we can nevertheless use eqs. (37),(39) as an approximation to a realistic physical situation in which instead of the δ -distribution we have a distribution due to an extended source.

The 6D Ricci scalar can be written as

$$R^{(6)} = R^{(4)} + \text{extrinsic curvature term}_{5,6}, \quad (40)$$

where the subscript 5,6 means that the extrinsic curvature is due to the presence of the 5th and 6th dimension. Instead of performing such ADM-like 2+4 decomposition, we will follow an easier procedure. We will consider a 1 + 5 decomposition in which case we have

$$R^{(6)} = R^{(5)} + \text{extrinsic curvature term}_6 \quad (41)$$

If there exist suitable isometries in the 6D space $M_{2,4}$, and if we choose a suitable 5D subspace $M_{2,3}$, then the extrinsic curvature terms in eq. (41) can be neglected. The 5D Ricci scalar, in turn, can also be decomposed in analogous way:

$$R^{(5)} = R^{(4)} + \text{extrinsic curvature term}_5 \quad (42)$$

In particular, let us consider the ADM-like 1+4 decomposition $M_{2,3} = \mathbb{R} \times M_{1,3}$, where $M_{1,3}$ is spacetime. Then the 5D metric can be decomposed as

$$G_{MN} = \begin{pmatrix} N_\mu N^\mu + N^2, & N_\mu \\ N_\nu, & g_{\mu\nu} \end{pmatrix}, \quad \begin{matrix} M, N = 0, 1, 2, 3, 5 \\ \mu, \nu = 0, 1, 2, 3 \end{matrix} \quad (43)$$

where the indices M, N now assume five values only, and $N = 1/\sqrt{G^{55}}$. The inverse metric is

$$G^{MN} = \begin{pmatrix} 1/N^2, & N^\mu/N^2 \\ N^\nu/N^2, & g^{\mu\nu} + N^\mu N^\nu/N^2 \end{pmatrix} \quad (44)$$

Of course, the meaning of N as an index is different from the meaning of N as a quantity, in which case $N = 1/\sqrt{G_{55}}$.

The extrinsic curvature is

$$K_{\mu\nu} = \mathcal{D}_\nu n_\mu = \frac{1}{2N} \left(D_\nu N_\mu + D_\mu N_\nu - \frac{\partial g_{\mu\nu}}{\partial \tau} \right), \quad \tau \equiv x^5. \quad (45)$$

Here \mathcal{D}_ν is the 5D covariant derivative, D_ν the 4D covariant derivative, and n_M the normal to $M_{1,3}$.

We see that the 4D metric $g_{\mu\nu}$ depends not only on four spacetime coordinates x^μ , but also on an extra parameter τ .

Introducing

$$p^{\mu\nu} = \sqrt{-g} (K g^{\mu\nu} - K^{\mu\nu}), \quad (46)$$

where $g \equiv \det g_{\mu\nu}$, and $K \equiv g^{\mu\nu} K_{\mu\nu}$, we can write the 5D action in the ‘phase space’ form:

$$I_G[g_{\mu\nu}, p^{\mu\nu}, N, N^\mu] = \frac{1}{16\pi\mathcal{G}} \int d\tau d^4x [p^{\mu\nu} \dot{g}_{\mu\nu} - N \mathcal{H}(g_{\mu\nu}, p^{\mu\nu}) - N_\mu \mathcal{H}^\mu(g_{\mu\nu}, p^{\mu\nu})], \quad (47)$$

where

$$\mathcal{H} = 2\sqrt{-g} N^2 G^{55} = \sqrt{-g} (R^{(4)} + K^2 - K^{\mu\nu} K_{\mu\nu}) \quad (48)$$

$$\mathcal{H}^\mu = 2\sqrt{-g} N G^{5\mu} = 2 D_\nu p^{\mu\nu} \quad (49)$$

The terms with extrinsic curvature in eq. (48) can be expressed in terms of $p^{\mu\nu}$:

$$K^2 - K^{\mu\nu} K_{\mu\nu} = \frac{1}{-g} \left(\frac{p^2}{D-1} - p^{\mu\nu} p_{\mu\nu} \right), \quad (50)$$

where $D = g_{\mu\nu} g^{\mu\nu} = 4$ and $p \equiv g_{\mu\nu} p^{\mu\nu} = \sqrt{-g} (D-1) K$.

Here $p^{\mu\nu}$ are the canonical momenta conjugated to the 4D metric $g_{\mu\nu}$, whilst N and N_μ are Lagrange multipliers for the constraints

$$\mathcal{H} = 0, \quad (51)$$

$$\mathcal{H}^\mu = 0. \quad (52)$$

Upon quantization, $g_{\mu\nu}$ and $p^{\mu\nu}$ become operators that can be represented as

$$g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad p_{\mu\nu} \rightarrow -i \frac{\delta}{\delta g_{\mu\nu}} \quad (53)$$

The ‘Hamiltonian’ constraint, $\mathcal{H} \approx 0$, becomes the Wheeler–DeWitt equation:

$$\left[-\frac{1}{\sqrt{-g}} \left(\frac{g_{\mu\nu} g_{\alpha\beta}}{D-1} - g_{\mu\alpha} g_{\nu\beta} \right) \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} + \sqrt{-g} R^{(4)} \right] \Psi[g_{\mu\nu}] = 0, \quad D = 4. \quad (54)$$

Now the wave function(al) depends on 4-geometry, represented by a spacetime metric $g_{\mu\nu}(x^\mu)$. In this theory we have no problem of spacetime. We also have no problem of time, if by ‘time’ we understand the coordinate time $t \equiv x^0$.

However, the evolution parameter τ has disappeared from the quantized theory. There is no τ in the wave functional equation (54). Now we have the problem of τ . One possibility is to take the position that this is not a problem. It is important that we do not have the problem of $t \equiv x^0$, whereas missing τ is not a problem at all!

Another possibility is to bring τ into the game by considering matter degrees of freedom. In our approach the latter degrees of freedom are described by coordinates of Clifford space, one of them being interpreted as τ . To describe matter configurations, we have to consider also the matter part of the action.

As a model we consider the action (37) in which $R^{(6)}$ is replaced with $R^{(5)}$, and d^6x with d^5x , the indices being now $M, N = 0, 1, 2, 3, 5$. The gravitational part we then replace by the equivalent phase space action (47). The matter part of the action we also replace by the phase space form:

$$I_m = \int d\sigma \left(P_M \dot{X}^M - \frac{\alpha}{2} (G_{MN} P^M P^N - M^2) \right) \quad (55)$$

Splitting the metric according to (43), we have

$$I_m = \int d\sigma \left(P_M \dot{X}^M - \frac{\alpha}{2} [g_{\mu\nu} (P^\mu + N^\mu P^5) (P^\nu + N^\nu P^5) + N^2 P^5 P^5 - M^2] \right) \quad (56)$$

To cast the matter part into a form comparable to the gravitational part of the action, we insert the integration over $\delta^5(x - X(\sigma))d^5x$, which gives identity. In both parts of the action, I_m and I_G , now stands the integration over d^5x . Recall that we identified $x^5 \equiv \tau$.

Varying the total action

$$I = I_G + I_m \quad (57)$$

with respect to α , N and N^μ , we obtain the constraints

$$\delta\alpha : \quad g_{\mu\nu} (P^\mu + N^\mu P^5) (P^\nu + N^\nu P^5) + N^2 P^5 P^5 - M^2 = 0, \quad (58)$$

$$\delta N : \quad \mathcal{H} + \kappa \int d\sigma \alpha N \delta^5(x - X(\sigma)) P^5 P^5 = 0, \quad (59)$$

$$\delta N^\mu : \quad \mathcal{H}_\mu - \kappa \int d\sigma \alpha \delta^5(x - X(\sigma)) g_{\mu\nu} (P^\nu + N^\nu P^5) P^5 = 0. \quad (60)$$

where \mathcal{H} and \mathcal{H}^μ are given in eqs. (48),(49), and $\kappa \equiv 16\pi\mathcal{G}$. We can write \mathcal{H} compactly as

$$\mathcal{H} = \mathcal{G}_{\mu\nu\alpha\beta} p^{\mu\nu} p^{\alpha\beta} + \sqrt{-g} R^{(4)}, \quad (61)$$

with the metric

$$\mathcal{G}_{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}} \left[\frac{g_{\mu\nu} g_{\alpha\beta}}{D-1} - \frac{1}{2} (g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha}) \right], \quad D = 4. \quad (62)$$

In a quantized theory, the constraints (58)–(60) become operator equations acting on a state vector. The constraint (58) can be put straightforwardly into its quantum version by replacing $P_\mu \rightarrow \hat{P}_\mu = -i\partial_\mu$, $P_5 \rightarrow \hat{P}_5 = -i\partial_5$. So we have

$$\left[g_{\mu\nu} (\hat{P}^\mu + N^\mu \hat{P}^5) (\hat{P}^\nu + N^\nu \hat{P}^5) + N^2 \hat{P}^5 \hat{P}^5 - M^2 \right] \Psi = 0. \quad (63)$$

But the constraints (59), (60), because of the δ -distribution, are not suitable for a direct translation into their corresponding quantum equivalents. Usually, for a quantum description of gravity in the presence of matter, one does not take the matter action in the form (56). Instead, one takes for I_m an action for, e.g., a scalar or spinor field, and then attempt to quantize the total action (57) following the established procedure of quantum field theory. Here I would like to point out that one can nevertheless start from the action (56) and use all the constraints (58)–(60).

Let us consider the Fourier transform of the constraint (59), the zero mode being given by the integral

$$\int d^5x \mathcal{H} = -\kappa \int \alpha d\sigma N (P^5)^2 \quad (64)$$

Writing $d^5x = d^4x dx^5$ and introducing $H = \int d^4x \mathcal{H}$, we have

$$\int dx^5 H = -\kappa \int \alpha d\sigma N (P^5)^2, \quad (65)$$

or

$$dx^5 H = -\kappa \alpha d\sigma N (P^5)^2, \quad (66)$$

from which it follows

$$\frac{1}{\alpha} \frac{dX^5}{d\sigma} H = -\kappa N (P^5)^2. \quad (67)$$

Here we have replaced the coordinate x^5 , denoting a point in the $5D$ manifold, with the coordinate X^5 , denoting a point on the worldline. Using the equation of motion (resulting from varying the action (55) with respect to P^M),

$$P^M = \frac{\dot{X}^M}{\alpha}, \quad (68)$$

where $\dot{X}^M \equiv dX^M/d\sigma$, we find that $P^5 = \dot{X}^5/\alpha$. Using the latter expression in eq. (67), we obtain

$$H = -\kappa N P^5. \quad (69)$$

Similarly, from the constraint (60) we obtain

$$H_\mu = \kappa g_{\mu\nu}(P^\nu + N^\nu P^5), \quad (70)$$

where $H^\mu = \int d^4x \mathcal{H}^\mu$. Let us now use the relations $P^M = G^{MN}P_N$ and $P_M = G_{MN}P^N$ with the metrics (43),(44), and rewrite eqs.(69),(70) into the form with covariant components of momenta P_μ , P_5 :

$$H = -\kappa \frac{1}{N}(P_5 + N^\mu P_\mu), \quad (71)$$

$$H_\mu = \kappa P_\mu. \quad (72)$$

The above result is nothing but a manifestation of the fact that the integration of a stress-energy tensor over a certain hypersurface gives momentum. Here momentum is $P_M = (P_\mu, P_5)$. Using (49), eq. (72) can be rewritten as

$$2 \int_\Omega d^4x D_\nu p^{\mu\nu} = 2 \int_B d\Sigma_\nu p^{\mu\nu} = \kappa p^\mu, \quad p^\mu \equiv g^{\mu\nu} P_\nu, \quad (73)$$

where B is the boundary of a region Ω in the 4-space, and $d\Sigma_\nu$ is an element of the boundary surface. The relation (73) is analogous to the Gauss law in electrodynamics. Bear in mind, that the momentum P^M points along a worldline $X^M(\sigma)$, $M = (\mu, 5)$, which intersects 4D spacetime in one point. Therefore, the integral in eq. (73) is different from zero only when the 3-surface B embraces the intersection point.

For the Lagrange multipliers we can choose $N = 1$ and $N^\mu = 0$, which simplifies eqs. (71) and (58) into

$$H = -\kappa P_5, \quad (74)$$

$$g^{\mu\nu} P_\mu P_\nu + P_5 P_5 - M^2 = 0. \quad (75)$$

It is now straightforward to consider the quantum versions of the constraints (74),(72) together with the constraint (75). We have

$$\left(g^{\mu\nu} \frac{\partial^2}{\partial X^\mu \partial X^\nu} + \frac{\partial^2}{\partial \tau^2} + M^2 \right) \Psi = 0 \quad (76)$$

$$\frac{1}{\kappa} \int d^4x \left(-G_{\mu\nu\alpha\beta} \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} + \sqrt{-g} R^{(4)} \right) \Psi = i \frac{\partial}{\partial \tau} \Psi, \quad \tau \equiv X^5. \quad (77)$$

$$\frac{2}{\kappa} \int d^4x D_\nu \left(-i \frac{\delta}{\delta g_{\mu\nu}} \right) \Psi = -i \frac{\partial}{\partial X^\mu} \Psi \quad (78)$$

The latter equations impose the operator constraints on a quantum state that is represented by $\Psi[\tau, X^\mu, g_{\mu\nu}(x^\mu)]$ which depends on the particle's coordinates X^μ , the fifth coordinate $X^5 \equiv \tau$, and the spacetime metric $g_{\mu\nu}(x^\mu)$. In other words, Ψ is a

function of τ , X^μ , and a functional of $g_{\mu\nu}(x^\mu)$. Eq. (77) is just like the Schrödinger equation, with τ as evolution parameter. Therefore, the “problem of τ ” does not exist in this quantum model for a point particle coupled to a gravitational field. Had we performed a split from six to four dimensions (and not from five to four as we did in this section), then in eq. (76), instead of ∂_τ^2 , we would have $\partial_\lambda \partial_\tau \sim \Lambda \partial_\tau$ (see sec. 2), so that eq. (76) would become the Stueckelberg equation.

The system (76)–(78) describes at once a Klein-Gordon wave function for a relativistic particle, and the wave functional for a gravitational field. It is only an incomplete description of the physical system. A complete description would require to take into account the infinite set of constraints due to all Fourier modes of the the constraints (58)–(60).

5 Discussion and conclusion

We have shown how the Stueckelberg equation for a relativistic point particle comes from a 6-dimensional space, $M_{2,4}$, with signature (2,4), that is $(+ - - - - +)$. Two extra dimensions, one time like and one space like, are necessary, because then in the equation we obtain the first derivative of the wave function with respect to a Lorentz, $SO(1,3)$, invariant parameter τ which is identified with the fifth coordinate X^5 .

An argument in favor of such 6D space comes from the works on the two time (2T) physics considered by Bars et al. [5]. The phase space action (16) for a point particle is a particular case of a more general action that is invariant under local $Sp(2)$ transformations between coordinates and momenta. In such theory there are three Lagrange multipliers associated with three constraints, which cannot be satisfied in 4D spacetime $M_{1,3}$. They can be satisfied in 6D space $M_{2,4}$. Since the theory by Bars et al. [5] is based on very strong foundations, we can conclude that such 6D space is a reasonable substitute for 4D spacetime. It enables to formulate the 2T physics on the one hand, and the Stueckelberg theory on the other hand. A possible deeper relationship between the two theories has yet to be explored.

There exists another direction of research, which is based on the concept of configuration space, i.e., the space of possible matter configurations. An example of such space is the 16D space of oriented r -volumes, associated with extended objects, e.g., branes. We call it Clifford space, C , because it is a manifold whose tangent space at any point is Clifford algebra $Cl(1,3)$. If we define the metric according to eq. (34), then the signature of C is (8,8). A subspace of C is $M_{2,4}$. Therefore, if we adopt the concept of Clifford space, C , we do not need to postulate extra dimensions of spacetime, in order to have the 6D space formulation of the Stueckelberg theory and of the 2T physics. Four dimensions of C can be identified with the four dimensions of spacetime, whilst the remaining 12 dimensions of C are associated with the intrinsic

configurations of matter living in the 4-dimensional spacetime.

We have considered the general relativity in Clifford space, more precisely in the 6D subspace with signature (2,4). The action contains the Einstein-Hilbert term which is a functional of the metric only, and a matter term, which is a functional of matter degrees of freedom coupled to the metric. As a model we have considered a point like source. We have performed the ADM decomposition of a 5D subspace into spacetime $M_{1,3}$ and a part due to the 5th dimensions, $x^5 \equiv \tau$. The action gives the mass shell constraint in 5-dimensions, and the constraints that generalize the Hamiltonian and momentum constraints of the canonical gravity, with the extra terms due to the presence of the point particle source. After quantization those constraints become the operator constraints acting on a state, that can be represented as a functional of the spacetime metric $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, a function of the particle coordinates X^μ , and the fifth coordinates, τ , which has the role of the Stueckelberg evolution parameter. In the Stueckelberg theory the ‘true’ time is the Lorentz, $SO(1,3)$, invariant evolution parameter τ , and not the coordinate $x^0 \equiv t$. Since such parameter occurs in the wave function(al) for the gravitational field, we conclude that there is no ‘problem of time’ in this theory.

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